ON CIRCULAR DESIGN WITH 3 AND 4 PLOT BLOCKS

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1. Introduction

Circular designs were first defined by Das (1960). A circular design with v treatments in blocks of size k can be obtained by slightly modifying his definition, *i.e.*, by developing the initial block, $1, 1+d, \cdots 1+(k-1)d$, mod (v) where d is any number prime to v. As the efficiency of a design does not depend on d, the value of d will be taken as 1 and this will simplify the construction and analysis of the design. For example, if $k = 3$ the design will have $b\,(= v)$ blocks each of size 3, each treatment replicated 3 times. These designs have the special advantage that they are available for any number of varieties and do not involve any problem of construction (Kempthorne, 1952). As it has been indicated by Das (1960), these designs are partially balanced incomplete block design with $(v-1)/2$ or $v/2$ associate classes according as v is odd or even. Evidently the analysis of these designs following the method of P.B.I.B. designs (Bose, 1939 and Rao, 1947) will be very complicated due to very large number of associate classes. But Das has shown that the normal equations corresponding to these designs can be solved without much inconvenience, as the number of normal equations can be reduced to $(k-1)$ equations, whatever v may be. Though he indicated in a general way the method of analysis for any k, specific results were given by him only in the case of $k = 2$. When $k = 3$, he indicated that suitable tables have to be prepared by consulting which the solutions of the equations can be obtained. In the present paper the complete analysis of designs with $k = 3$ and 4 have been presented together with the preparation of necessary tables. Moreover, it has been shown that by modifying the definition of such designs as stated afterwards, some new series of designs can be obtained which give in general more efficient estimates than those obtained through circular designs with the same block size and replication. Further the circular designs with $k = 2$ have the limitation that the number of replications must be 2 and the degrees of freedom for error mean square is always unity. It has been attempted to combine several designs with $k = 2$ and get fresh designs which do not suffer from any of the limitations (Kempthorne, 1953). As the method of analysis follows more or less on the same lines as suggested by Das, it has not been described any further but the final results for the intra-block analysis have been presented below separately for each case.

2. Particular Cases

2.1. *Case 1—When $k = 2$*

Though this case has been presented by Das, it has been presented here only for the sake of completeness.

The design is obtained by developing the initial block (1, 2) mod (v) or the initial block with any constant difference. The solution for t_i for this design is

$$vt_i = \sum_{u=1}^{p} (p - u+1)(p - u+2) Q_i^{u-1} \quad \left(v \text{ is odd and } p = \frac{v-1}{2} \right)$$

$$= \sum_{v=1}^{p} (p - u + 1)^2 Q_i^{u-1} \quad \left(v \text{ is even and } p = \frac{v}{2} \right)$$

where Q_a is the adjusted total of the a-th treatment and $Q_i^u = Q_{i+u} + Q_{i-u}$, $(i + u)$ and $(i - u)$ being taken mod v wherever necessary and Q_i^0 is to be taken as Q_i only. These definitions of Q_i^u, etc., have been used throughout the paper. The different variances of differences between treatment pairs are given by

$$Var\ (t_i - t_{i+u}) = \frac{2u(v-u)}{v} \sigma^2$$

$$u = 1, \cdots p$$

where σ^2 is the error variance. Efficiency factor as worked out from the average variance $= 3/(v+1)$.

No table is necessary in this case.

2.21. *Case 2a—When $k = 3$ and v is odd*

The design can be obtained by developing the initial block (1, 2, 3) mod v.

The solution for t_i for the design is given by the expression

$$4vt_i = \sum_{u=1}^{p} (p-u+1)(p-u+2)\, Q_i^{u-1} - \frac{2v}{\beta_1} \sum_{u=1}^{p} U_{p-u} Q_i^{u-1} .$$

where

$$\beta_1 = -2U_{p-1} + 5U_{p-2} - 2U_{p-3} - U_{p-4} \text{ and } p = \frac{v-1}{2} .$$

The different U's can be obtained from the recurrence relation

$$U_u = -2U_{u-1} + 6U_{u-2} - 2U_{u-3} - U_{u-4};$$

the initial values of U's being

$$U_0 = 1,\ U_1 = -3,\ U_2 = 12, \text{ etc.}$$

The variances of differences between treatment pairs are given by

$$\text{Var}\,(t_i - t_{i\pm u}) = \frac{1}{2v}\left\{u(v-u) - \frac{2v}{\beta_1}(U_{p-1} - U_{p-u-1})\right\}\sigma^2$$

$$u = 1,\ 2, \cdots, p.$$

When $u = p_0$, U_{p-u-1} is to be taken as zero. The values of U's and β_1, for v up to 49 have been tabulated and presented in Table I.

Efficiency factor as obtained from the average variance

$$= \frac{4vp}{vp(p+1) - \frac{6v}{\beta_1}\left(pU_{p-1} - \sum_{u=1}^{p} U_{p-u-1}\right)}$$

and tabulated values have been given in the table for different values of v up to 30.

2.22. *Case 2b—When $k = 3$; and v is even*

The solution for t_i for the designs is given by the expression

$$4vt_i = \frac{3}{2}\sum_{u=1}^{p} (p-u+1)^2 Q_i^{u-1} - \frac{2v}{\beta_1}\sum_{u=1}^{p} U_{p-u} Q_i^{u-1} .$$

where β_1, U_u and variance expressions are the same as given above except that the initial values in the recurrence relation are different, *viz.*, $U_0 = 1$, $U_1 = -2$, $U_2 = 9$, etc.

TABLE I

Values of U_r and β_1 which appear in the solution of the treatment effects in designs with block size 3

r	U_r	v	β_1	E.F.
0	1	5	11	0·8148
1	–3	7	–41	0·6833
2	12	9	153	0·5862
3	–44	11	–571	0·5152
4	165	13	2,131	0·4549
5	–615	15	–7,953	0·4088
6	2,296	17	29,681	0·3787
7	–8,568	19	–1,10,771	0·3398
8	31,977	21	4,13,403	0·3133
9	–1,19,339	23	–15,42,841	0·2906
10	4,45,380	25	57,57,961	0·2710
11	–16,62,180	27	–2,14,89,003	0·2528
12	62,03,341	29	8,01,98,015	0·2378
13	–2,31,51,183	31	–29,93,03,201	..
14	8,64,01,392	33	1,11,70,14,609	..
15	–32,24,54,384	35	–4,16,87,55,715	..
16	1,20,34,16,145	37	15,55,80,08,539	..
17	–4,49,12,10,195	39	–58,06,32,78,153	..
18	16,76,14,24,636	41	2,16,69,51,04,121	..
19	–62,55,44,88,348	43	–8,08,71,71,38,334	..
20	2,33,45,65,28,757	45	30,18,17,34,55,203	..
21	–8,71,27,16,26,679	47	–1,12,63,97,66,58,481	..
22	32,51,62,99,77,960	49	4,20,37,73,31,84,724	..
23	–1,21,35,24,82,85,760	..	..	..
24	4,52,89,36,31,62,681	..	..	..

Efficiency factor

$$= \frac{4v(v-1)}{p(v^2-1) - \frac{6v}{\beta_1}\left\{(v-1)U_{p-1} - 2\sum_{u=1}^{p-1} U_{p-u-1}\right\}}$$

Tabulated values for β_1 and U's have been presented in Table II.

2.31. *Case 3a—When $k = 4$ and v is odd*

The design can be obtained by developing the initial block (1, 2, 3, 4) mod v.

The solution for t_i for the design is given by

$$10vt_i = \sum_{u=1}^{p} (p-u+1)(p-u+2)\, Q_i^{u-1}$$

$$+ \frac{2v(\beta_2 - 4\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p-1} U_{p-u-1}\, Q_i^{u-1}$$

$$+ \frac{2v(4\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p} V_{p-u}\, Q_i^{u-1}$$

where

$$\beta_1 = -3U_{p-2} + 10U_{p-3} - 4U_{p-4} - 2U_{p-5} - U_{p-6}.$$

$$\gamma_1 = U_{p-1} - U_{p-3}.$$

$$\beta_2 = -3V_{p-1} + 10V_{p-2} - 4V_{p-3} - 2V_{p-4} - V_{p-5}.$$

$$\gamma_2 = V_p - V_{p-2}.$$

$$U_u = -2U_{u-1} - 3U_{u-2} + 12U_{u-3} - 3U_{u-4} - 2U_{u-5} - U_{u-6}.$$

$$V_u = -2V_{u-1} - 3V_{u-2} + 12V_{u-3} - 3V_{u-4} - 2V_{u-5} - V_{u-6}.$$

The initial values of U's and V's are

$$U_0 = 1,\ U_1 = -3,\ U_2 = 3,\ U_3 = 15, \text{ etc.}$$

$$V_0 = 1,\ V_1 = 1,\ V_2 = -8,\ V_3 = 24, \text{ etc.}$$

The different variances of difference between treatment pairs are given by

TABLE II

Values of U_r and β_1 which appear in the solution of the treatment effects in designs with block size 3

r	U_r	v	β_1	E.F.
0	1	6	30	0·7435
1	−2	8	−112	0·6312
2	9	10	418	0·5498
3	−32	12	−1,560	0·4820
4	121	14	5,820	0·4306
5	−450	16	−21,728	0·3891
6	681	18	81,090	0·3548
7	−6,272	20	−3,00,632	0·3262
8	23,409	22	11,20,438	0·3014
9	−86,362	24	41,83,120	0·2845
10	3,24,041	26	1,56,10,042	0·2612
11	−12,06,800	28	−5,82,59,048	0·2460
12	45,07,161	30	21,74,24,150	0·2257
13	−1,68,16,842	32	−81,14,39,552	..
14	6,27,66,209	34	3,02,83,32,058	..
15	−23,42,40,992	36	−11,30,18,90,680	..
16	87,42,05,761	38	42,17,92,78,662	..
17	−3 26,25,73,050	40	−1,57,41,50,25,968	..
18	1217,60,96,441	42	5,87,48,08,73,210	..
19	−45,44,18,01,712	44	−21,92,50,84,68,872	..
20	1,69,59,11,22,409	46	81,82,55,30,00,278	..
21	−6,32,92,26,74,922	48	−3,05,37,70,35,34,240	..
22	23,62,09,95,91,281	..	..	..
23	−88,15,47,56,75,200	..	..	..
24	3,28,99,80,31,25,521	..	..	..

$$\text{Var}\,(t_i - t_{i\pm u})$$

$$= \frac{1}{5v}\left\{u(v-u) + \frac{2v(\beta_2 - 4\gamma_2)}{\beta_1\gamma_2 - \beta_2\gamma_1}(U_{u-2} - U_{p-u-2}) + \frac{2v(4\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)}(V_{p-1} - V_{p-u-1})\right\}\sigma^2$$

$$u = 1, \cdots p.$$

Tabulated values of U's, V's, $(4\gamma_1 - \beta_1)\,v$, $(\beta_2 - 4\gamma_2)\,v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ and efficiency factor have been presented in Table III.

2.32. *Case 3b—When $k = 4$ and v is even*

The solution for t_i is

$$10\,vt_i = 2\sum_{u=1}^{p}(p-u+1)^2 Q_i^{u-1} + \frac{2v(\beta_2 - 4\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \times \sum_{u=1}^{p-1} U_{p-u-1}\,Q_i^{u-1} + \frac{2v(4\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)}\sum_{u=1}^{p} V_{p-u}Q_i^{u-1}$$

The initial values for U's and V's are:

$$U_0 = 1,\ U_1 = -2,\ U_2 = 0,\ U_3 = 18,\ \text{etc.}$$

$$V_0 = 1,\ V_1 = 1,\ V_2 = -5,\ V_3 = 16,\ \text{etc.}$$

Tabulated values for U's, V's, $(\beta_2 - 4\gamma_2)\,v$, $(4\gamma_1 - \beta_1)\,v$ and $(\beta_1\gamma_2 - \beta_2\gamma_1)$ have been presented in Table IV together with E.F.

3. Some Modified Circular Designs

All the above designs have been developed from only one initial block giving as many replications as the block size. Sometime particularly when $k = 2$, it may be necessary to have more replications keeping the block size the same. This is possible by taking more than one initial block and getting the designs by developing them. Such designs have been discussed by Kempthorne (1953). The initial blocks can be chosen in various ways. By just repeating one initial block, it is possible to get such a design. But the efficiency of the designs can be increased by taking different initial blocks. A series of designs can be obtained by taking different initial blocks when $k = 2$ and 3 and have been presented below.

TABLE III

Values of U_r, V_r, $(4\gamma_1 - \beta_1)$, v, $(\beta_2 - 4\gamma_2)\, v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ and E.F. which appear in the solution of the treatment effects in design with block size 4 and odd number of treatments

r	U_r	V_r	v	$(4\gamma_1-\beta_1)v$	$(\beta_2-4\gamma_2)v$	$(\beta_1\gamma_2-\beta_2\gamma_1)$	E.F.
0	1	1	5	−45	215	48	0·9800
1	−8	1	7	−77	−434	377	0·8249
2	3	−8	9	1,037	−1,170	3,145	0·7711
3	15	24	11	−3,509	10,802	26,269	0·7026
4	−76	−15	13	2,327	−31,486	2,19,413	0·6443
5	154	−143	15	30,375	6,750	18,32,625	0·6059
6	102	640	17	−1,48,392	3,16,574	1,53,06,833	0·5516
7	−1,650	−1,088	19	2,71,111	−13 35,358	13,01,90,682	0·5206
8	5,043	−1,455	21	4,60,635	20 06.130	1,06,78,46,845	0·4816
9	−4,233	14,289	23	−45,43.397	56,52,526	8,91,90,94,697	0·4523
10	−26,999	−38,888	25	1,30,61,475	−4,18,96,050	74.49,59,36,025	0·4224
11	1,31.805	19,576	27	−58,44,285	10,62,15,570	6,22,22,06,03,405	0·4044
12	−2.45 340	2,54,881	29	10,41,25,921	−16,18,142	51,97,04,16,10,021	0·3838
13	−2.24.460	−10,74,015	31	45,36,67,919	−1,01,24,07,362	4,32,35 93,61,00.481	..
14	28.51,020	17,07,840	33	−73,55,35,185	3,86,53,33.870	36 35,59,93,13,55,105	..
15	−83 09,924	28,69,696	35	−1,49,13,52,065	−5,09,78,64,030	3,02,82,49,41,50,80,565	..
16	58,72 677	−2,45,15,999	37	12,52,01,50.243	−1,66,73,25,02,894	25,21,90,76,50,14,36,557	..
17	4,84.28,913	6.36,09,697	39	−33,26,34,05,565	1,11,25,57.98,030	2,11,25 90,02,90,02,97,945	..
18	−22,20,33,745	−2,24,65,496	41	9,04,19,81,371	−2,60,97 52,84,091	20,77,67,81,38,03,72,88,605	..
19	38,59 68,854	−45,10,41,000	43	2,72,44,86,04,547	−46,80,22,2[illegible],046	*	..
20	46,59,68,854	1,79,88,95,377	45	−11,01,42,76,01 625	25,67,40,13,73,250	*	..
21	−4,91,15,40,018	−2,65,89,19,535	47	16.22,21.89,92,887	−91,51 88,36,00,866	*	..
22	13 65,35 45,010	−5,52,66,96,208	49	39,75,65,93,80,231	5,62,98,22,35,41,910	*	..
23	−7,75,14,00,342	41,95,12,39,680		..	..	..	..

* Means that figures are too large in these entries.

TABLE IV

Values of U_r, V_r, $(4\gamma_1 - \beta_1)v$, $(\beta_2 - 4\gamma_2)v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ and E.F. which appear in the solutions of the treatment effect in designs with block size 4 (Even number of treatments)

r	U_r	V_r	v	$(4\gamma_1-\beta_1)v$	$(\beta_2-4\gamma_2)v$	$(\beta_1\gamma_2-\beta_2\gamma_1)$	E.F.
0	1	1	6	−120	−234	261	0·9016
1	−2	1	8	832	−704	2,176	0·8095
2	0	−5	10	−1,410	6,480	18,180	0·7537
3	18	16	12	−1,680	−18.960	1,51,840	0·6723
4	−63	−9	14	30,856	3,248	12,68,228	0·6185
5	76	−95	16	−1,07,264	1,97,888	1,11,81,824	0·5831
6	256	421	18	99,720	6,30,160	8,84,75,140	0·5323
7	−1,548	−704	20	7,24,080	12,48,240	74,54,05,920	0·4979
8	3,393	−995	22	−38,63,288	35,97,616	6,22,39,14,196	0·4668
9	810	9,441	24	77,98,080	−2,65,87,200	51,98,38,35,200	0·4393
10	−31,232	−25,649	26	79,56,104	6.73 83,472	4,34,18,80,01,972	0·4155
11	1,04,806	12,640	28	−10,65,96,112	−1,52,656	36,26 51,81,59,744	0·4001
12	−1,13,535	1,68,931	30	33,13,17,000	−64,88,17,200	3,02,90,16,80,49,300	0·3657
13	−4,69,800	−7,09,199	32	−24,49,43,872	2,47,37,46,944	25,29,95,90,08,94,461	..
14	26,26,560	11,22,345	34	−2,20,65,66,136	−3,24,75,38,288	2,11,31,25,50,05,32,072	..
15	−54,58,904	19,14,016	36	10,64,78,05,680	−1,09,60,84,480	17,64,96,90,63,22,65,920	..
16	−24,37,247	−1,62,11,879	38	−19,40,91,45,840	7,16,46,71,66,556	1,46,92,67,28,69,86,46,756	..
17	5,43,01,590	4,19,26,945	40	−25,63,31,24,160	−1,67,69,27,64,480	12,29,63,83,86,15,99,70,560	..
18	−17,26,85,232	−1,43,67,629	42	2.74,94,10,15,720	2,15,62,15,65,684	*	..
19	16,38,72.522	−29,88,65,664	44	−2,93,41,24,35,024	16,62,00,47,89,808	*	..
20	85,75,32,721	1,18,76,42,815	46	4,48,01,22,34,216	−59,10.54,76,72,432	*	..
21	−4,43,14,77,164	−1,74,63,71,279	48	55,59,38.08,08,320	−68,88,55,49,49,120	*	..
22	8,66,87,16,192	−3,68,11,12,979	50	−2,47,54,46,71,39,800	2,80,26,79,05,87,600	*	..
23	6,04,68,43,068	27,73,64,58,880	..	..	..	*	..
24	−94,00,52,18,656	−68,33,68,63,659	..	..	..	*	..

* Entries in these columns are too large.

3.11. *Case* $4a$—$k = 2$, $r = 4$ (v is odd)

The layout of the design is obtained by developing the initial blocks, *viz.*, (1, 2) and (1, 3), mod v.

The solution for t_i for the design is given by the expression

$$(5vt_i) = \sum_{u=1}^{p} (p - u + 1)(p - u + 2) Q_i^{u-1} - \frac{2v}{\beta_1} \sum_{u=1}^{p} U_{p-u} Q_i^{u-1}.$$

where

$$\beta_1 = - U_{p-1} + 3U_{p-2} - U_{p-3} - U_{p-4}$$

and

$$U_u = - U_{u-1} + 4U_{u-2} - U_{u-3} - U_{u-4}.$$

$$U_0 = 1, \; U_1 = -2, \; U_2 = 6, \; U_3 = -15, \text{ etc.}$$

The different variances are given by

$$\text{Var}(t_i - t_{i \pm u}) = \frac{2}{5v} \left\{ u(v - u) - \frac{2v}{\beta_1} (U_{p-1} - U_{p-u-1}) \right\} \sigma^2$$

$$u = 1, \cdots p.$$

Tabulated values of U_u, β_1, and E.F. have been presented in Table V.

3.12. *Case* $4b$—$k = 2$, $r = 4$ (v is even)

The design is to be obtained from the same two initial blocks as when v is odd.

The solution for t_i is given by

$$5vt_i = \sum_{u=1}^{p} (p - u + 1)^2 Q_i^{u-1} - \frac{2v}{\beta_1} \sum_{u=1}^{p} U_{p-u} Q_i^{u-1}$$

where U's, β_1 and variance expressions are the same as in the above case except with the different initial values for U's, *i.e.*,

$$U_0 = 1, \; U_1 = -1, \; U_2 = 4, \; U_3 = -9, \; U_4 = 25, \text{ etc.}$$

Tabulated values of U_r, β_1 and E.F. have been presented in Table VI.

TABLE V

Values of U_r and β_1 which appear in the solution of the treatment effect in designs with block size two and with initial blocks (1, 2) *and* (1, 3) *and for odd and even number of treatment*

r	U_r	v	β_1	E.F.
0	1	5	5	0·6250
1	−2	7	−13	0·5417
2	6	9	34	0·4742
3	−15	11	−89	0·4246
4	40	13	233	0·3824
5	−104	15	−610	0·3476
6	273	17	1,597	0·3184
7	−714	19	−4,181	0·2938
8	1,870	21	10,946	0·2726
9	−4,895	23	−28,657	0·2542
10	12,816	25	75,025	0·2382
11	−33,552	27	−1,96,418	0·2240
12	87,841	29	5,14,229	0·2107
13	−2,29,970	31	−13,46,269	..
14	6,02,070	33	35,24,578	..
15	−15,76,239	35	−92,27,465	..
16	41,26,648	37	2,41,57,817	
17	−1,08,03,704	39	−6,33,45,986	..
18	2,82,84,465	41	16,55,80,141	..
19	−7,40,49,690	43	−4,33,49,44,337	..
20	19,33,61,606	45	1,13,49,03,170	..
21	−50,75,44,127	47	−2,97,12,15,073	..
22	1,32,87,67,776	49	7,77,87,42,049	..
23	−3,47,87,59,200	..	..	..
24	91,07,09,825	..	..	

TABLE VI

Values of U_r and β_1 which appear in the solution of the treatment effect in designs with block size two and with initial blocks (1, 2) *and* (1, 3) *and for odd and even number of treatment*

r	U_r	v	β_1	E.F.
0	1	6	−8	0·5769
1	−1	8	21	0·5069
2	4	10	−55	0·4492
3	−9	12	144	0·4024
4	25	14	−377	0·3642
5	−64	16	987	0·3324
6	169	18	−2,584	0·3056
7	−441	20	6,765	0·2828
8	1,156	22	−17,711	0·2631
9	−3,025	24	46,368	0·2459
10	7,921	26	−1,21,393	0·2309
11	−20,736	28	3,17,811	0·2175
12	54,289	30	−8,32,040	0·2057
13	−1,42,129	32	21,78,309	..
14	3,72,100	34	−57,02,887	..
15	−9,74,169	36	1,49,30,352	..
16	25,50,409	38	−3,90,88,169	..
17	−66,77,056	40	10,23,34,155	..
18	1,74,80,761	42	−26,79,14,296	..
19	−4,57,65,225	44	70,14,08,733	..
20	11,98,14,916	46	−1,83,63,11,903	..
21	−31,36,29,521	48	4,80,75,26,976	..
22	82,12,23,649	50	−12,58,62,69,025	..
23	−2,14,99,91,424	..	..	..
24	5,62,87,50,624	..	..	..

3.21. *Case* 5*a*—$k = 2$, $r = 6$

The layout of the design is obtained by developing the initial blocks, *viz.*, (1, 2), (1, 3) and (1, 4), mod v both when v is odd or even.

When v is odd

The solution for t_i for the design is given by

$$14vt_i = \sum_{u=1}^{p} (p-u+1)(p-u+2)\, Q_i^{u-1}$$

$$+ \frac{2v(\beta_2 - 3\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p-1} U_{p-u-1}\, Q_i^{u-1}$$

$$+ \frac{2v(3\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p} V_{p-u}\, Q_i^{u-1}$$

where

$$\beta_1 = -U_{p-2} + 5U_{p-3} - 2U_{p-4} - U_{p-5} - U_{p-6},$$
$$\gamma_1 = U_{p-1} - U_{p-3},$$
$$U_u = -U_{u-1} - U_{u-2} + 6U_{u-3} - U_{u-4} - U_{u-5} - U_{u-6},$$
$$U_0 = 1,\ U_1 = -2,\ U_2 = 1,\ U_3 = 7,\ \text{etc.},$$
$$\beta_2 = -V_{p-1} + 5V_{p-2} - 2V_{p-3} - V_{p-4} - V_{p-5},$$
$$\gamma_2 = V_p - V_{p-2},$$
$$V_u = -V_{u-1} - V_{u-2} + 6V_{u-3} - V_{u-4} - V_{u-5} - V_{u-6},$$
$$V_0 = 1,\ V_1 = 1,\ V_2 = -4,\ V_3 = 8,\ \text{etc.},$$

The different variances are given by

$$\text{Var}(t_i - t_{i\pm u})$$

$$= \frac{1}{7v}\left\{ u(v-u) + \frac{2v(\beta_2 - 3\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} (U_{p-2} - U_{p-u-2}) \right.$$

$$\left. + \frac{2v(3\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} (V_p - V_{p-u-1}) \right\} \sigma^2$$

$$u = 1, 2, \cdots p.$$

Tabulated values of U_u, V_u, $(3\gamma_1 - \beta_1)v$, $(\beta_2 - 3\gamma_2)v$ and $(\beta_1\gamma_2 - \beta_2\gamma_1)$ have been presented in Table VII.

TABLE VII

Values of U_r, V_r, $(3\gamma_1 - \beta_1)\,v$, $(\beta_2 - 3\gamma_2)\,v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ *which appear in the solutions of treatment effects in designs with block size two with initial blocks* (1, 2), (1, 3) *and* (1, 4) *for number of treatments*

r	U_r	V_r	v	$(3\gamma_1-\beta_1)\,v$	$(\beta_2-3\gamma_2)\,v$	$(\beta_1\gamma_2-\beta_2\gamma_1)$	E.F.
							0·6794
0	1	1	5	−25	95	13	0·5838
1	−2	1	7	−49	−98	49	0·5418
2	1	−4	9	360	−414	214	0·5072
3	7	8	11	−737	1,914	947	0·4754
4	−21	1	13	−169	2,886	4,187	0·4404
5	21	−85	15	4,920	−3,660	17,668	0·4140
6	42	64	17	−12,665	27,612	81,769	0·3976
7	−195	−48	19	7,087	−52,326	3,61,379	0·3767
8	292	−251	21	49,980	−882	15,97,106	0·3578
9	148	829	23	−1,71,833	2,82,302	70,58,377	0·3407
10	−1,652	−916	25	1,93,975	−7,46,050	3,11,94,361	0·3251
11	3,388	1,420	27	3,78,972	5,06,474	13,78,62,868	0·3109
12	−987	2,525	29	−19,90,241	24,93,594	60,92,82,227	..
13	−12,558	−12,131	31	32,81,814	−90,91,618	1,61,05,50,841	..
14	35,085	−3,576	33	20,48,574	1,11,36,642	12,26,80,61,908	..
15	−24,411	62,364	35	−2,12,05,975	1,64,68,410	58,06,58,91,653	..
16	−92,693	−1,36,763	37	4,33,83,721	−9,77,75,238	2,57,18,08,20,509	..
17	3,37,771	58,969	39	−58,94,733	17,07,52,530	11,84,30,05,56,619	..
18	−4,13,084	4,60,160	41	−19,38,00,727	3,89,55,166	50,19,91,75,30,783	..
19	4,78,961	−13,86,364	43	53,26,07,417	−93,33,45,702	2,21,85,20,89,98,261	..
20	30,00,690	13,57,993	45	−42,35,13,540	2,20,06,21,230	9,70,62,30,66,42,302	..
21	−52,20,900	28,04,761	47	−1,49,06,66,335	−1,11,43,51,106	49,84,09,16,28,45,855	..
22	−4,85,550	−1,28,63,304	49	5,84,00,93,273	−7,74,78,97,346	1,91,51,09,51,77,16,983	..
23	2,44,64,864	1,90,73,736	..	..	..	..	..
24	−5,72,13,359	1,01,86,345	..	..	..	..	..
25	3,27,34,366	−10,92,16,295	..	..	..		..

3.22. *Case 5b*—$k = 2,\ r = 6$ (v is even)

The solution for t_i is given by the expression

$$14vt_i = \sum_{u=1}^{p} (p - u + 1)^2 Q_i^{u-1} + \frac{2v(\beta_2 - 3\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p-1} U_{p-u-1} Q_i^{u-1}$$

$$+ \frac{2v(3\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p} V_{p-u} Q_i^{u-1}$$

where β's, γ's, U's, V's and variance expressions are the same as expressed above and the initial values in the difference equations are:

$$U_0 = 1,\quad U_1 = -1,\quad U_2 = -1,\quad U_3 = 8,\quad \text{etc.}$$

$$V_0 = 1,\quad V_1 = 1,\quad V_2 = -2,\quad V_3 = 5,\quad \text{etc.}$$

Tabulated values of quantities as indicated above have been presented in Table VIII.

3.3. *Case* 6—*When* $k = 3,\ r = 6$

This class of designs have been obtained by developing the initial blocks (1, 2, 3) and (1, 2, 4), mod v. The analysis of this type of design corresponds to the case $k = 4$, except with the minor changes as indicated.

The solution for t_i for the designs is obtained by multiplying the R.H.S. of the case 3 (*a*) and 3 (*b*) by 3/4. The variance will be 3/4th to that of $k = 4$ and the efficiency factor will be 8/9 times to that of $k = 4$.

4. Some Designs with Unequal Differences in the Initial Block

During the course of investigation it was seen that there is another class of designs, which are more efficient than the designs obtained in the previous section. The solution of designs with block size 3 with one initial block and of a design with block size 2 with two initial blocks is presented below.

4.1. *Case* 7

$k = r = 3$ with the initial block of the type (1, 2, 4) or (1, 3, 4) mod v. The analysis of this type of designs is the same as that of $k = 2, r = 6$, *i.e.*, Case Nos. 5 (*a*) and 5 (*b*) except for some minor changes.

TABLE VIII

Values of U_r, V_r, $(3\gamma_1 - \beta_1)\,v$, $(\beta_2 - 3\gamma_2)\,v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ *which appear in the solutions of treatment effects in designs with block size two with initial blocks* (1, 2), (1, 3) *and* (1, 4) *for even number of treatments*

r	U_r	V_r	v	$(3\gamma_1-\beta_1)v$	$(\beta_2-3\gamma_2)v$	$(\beta_1\gamma_2-\beta_2\gamma_1)$	E.F.
0	1	1	6	−72	−42	34	0·6273
1	−1	1	8	264	−216	144	0·5600
2	−1	−2	10	−270	1,010	638	0·5248
3	8	5	12	−960	−1,536	2,816	0·4909
4	−14	1	14	4,268	−1,[illegible]62	12,292	0·4557
5	0	−20	16	−6,672	14,2[illegible]6	55,008	0·4333
6	63	49	18	−6,588	−28,6[illegible]6	2,43,240	0·4095
7	−153	−27	20	55,060	−860	10,74,392	0·3869
8	97	−146	22	−1,12,002	1,57,102	47,48,258	0·3673
9	440	−481	24	6,912	−4,14,720	2,09,84,832	0·3491
10	−1,504	−527	26	−5,66,670	2,78,590	9,27,42,050	0·3364
11	1,735	−832	28	−15,28,604	14,04,340	40,98,71,672	0·3178
12	2,401	4,369	30	11,17,080	−51,01,680	1,81,14,19,808	0·3042
13	−13,545	−7,007	32	48,05,664	62,19,936	8,00,55,34,272	..
14	22,527	−2,162	34	−1,8[illegible],86,742	93,85,258	35,38,03,01,534	..
15	4,752	36,261	36	2,29,57,632	−5,[illegible]2,36,096	1,56,36,25,42,968	..
16	−1,11,182	−79,161	38	3,00,53,250	9,62,81,474	6,91,04,12,10,686	..
17	2,51,000	33,388	40	−19,01,27,400	−5,3[illegible],35,769	30,54,04,20,02,438	..
18	−1,22,689	2,68,129	42	34,15,91,544	−33,62,23,152	1,44,74,50,28,83,568	..
19	−8,09,137	−8,03,515	44	4,88,12,588	1,24,54,79,708	5,96,59,62,04,02,662	..
20	25,21,729	7,80,766	46	−1,77,82,7[illegible],173	−61,93,96,762	26,36,26,32,94,41,602	..
21	−25,93,296	16,41,025	48	4,30,42,68,288	−4,42,35,69,408	1,49,70,91,35,27,36,096	..
22	−48,00,384	−74,65,247	50	−2,42,64,28,650	1,42,48,36,77,552	4,44,90,07,70,86,28,162	..
23	2,32,14,880	−1,10,10,8[illegible]6	..	..	..	..	..
24	−3,[illegible]5,54,175	60,55,201	..	..	..	..	..
25	−1,55,72,305	−6,34,75,775	..	..	..	..	..

Solution of t_i is obtained by multiplying the R.H.S. of the solution in the case when $k = 2$ and $r = 6$ by 3/2 and the same change for the variance.

For this case, Tables VII and VIII have to be used for the analysis. Table (A) below indicates how the efficiency factor differs from the case when $k = 3$ with the initial block (1, 2, 3).

TABLE A

V	I	II
5	0·8148	0·9059
10	0·5498	0·6997
15	0·4088	0·5871
20	0·3262	0·5159
25	0·2710	0·4543
30	0·2257	0·4056

Col. I corresponds to the designs having the initial block of the type (1, 2, 3) and Col. II corresponds to the designs with the initial block of the type (1, 2, 4) or (1, 3, 4) mod v.

4.21. *Case* 8 *a—When* $k = 2$ *and* $r = 4$, $v = 2p + 1$, $b = 2v$

The designs are obtained by developing two initial blocks, *viz.*, (1, 3) and (1, 4) mod v. This class of designs are also partially balanced incomplete block design with p number of associate classes with the following parameters:

$$\lambda_i = 1 \ (i = 2, \ 3)$$

$$= 0 \ (i = 1, \ 4, \cdots p)$$

$$n_i = 2 \ (i = 1, \cdots p).$$

The solution for t_i for the design is given by

$$13vt_i = \sum_{u=1}^{p} (p - u + 1)(p - u + 2) Q_i^{u-1}$$

$$+\frac{2v(3\gamma_1-\beta_1)}{(\beta_1\gamma_2-\beta_2\gamma_1)}\sum_{u=1}^{p-1} U_{p-u-1}Q_i^{u-1}$$

$$+\frac{2v(\beta_2-3\gamma_2)}{(\beta_1\gamma_2-\beta_2\gamma)}\sum_{u=1}^{p} V_{p-u}Q_i^{u-1}$$

where

$$\beta_1 = 3U_{p-2} - U_{p-4} - U_{p-5} - U_{p-6}$$

$$\gamma_1 = U_{p-1} - U_{p-3}$$

$$U_u = -U_{u-1} + 4U_{u-3} - U_{u-5} - U_{u-6}$$

and

$$U_0 = 1,\ U_1 = -2,\ U_2 = 2,\ U_3 = 2,\ \text{etc},$$

$$\beta_2 = 3V_{p-2} - V_{p-3} - V_{p-4} - V_{p-5},$$

$$\gamma_1 = V_p - V_{p-2},$$

$$V_u = -V_{u-k} + 4V_{u-3} - V_{u-5} - V_{u-6}$$

and

$$V_0 = 1,\ V_1 = 1,\ V_2 = -3,\ V_3 = 6,\ \text{etc.}$$

The different variances are given by

$$\text{Var}(t_i - t_{i\pm u})$$

$$= \frac{2}{13v}\left\{u(v-u) + \frac{2v(3\gamma_1-\beta_1)}{(\beta_1\gamma_2-\beta_2\gamma_1)}(U_{p-2} - U_{p-u-2}) + \frac{2v(\beta_2-3\gamma_2)}{(\beta_1\gamma_2-\beta_2\gamma_1)}(V_{p-1} - V_{p-u-1})\right\}\sigma^2$$

$$u = 1, \cdots p.$$

Tabulated values of U_u, V_u, $(3\gamma_1-\beta_1)v$, $(\beta_2-3\gamma_2)v$, $(\beta_1\gamma_2-\beta_2\gamma_1)$ and E.F. have been presented in Table IX.

4.32. *Case* 8*b*—*When* $k=2$ *and* $r=4$, $v=2p$ (p is not a multiple of 3), $b=2v$

The primary parameters and recurrence relations are same as above. The solution for t_i for the design is given by

TABLE IX

Values of U_r, V_r $(3\gamma_1 - \beta_1)\,v$, $(\beta_2 - 3\gamma_2)\,v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ which appear in the solutions of treatment effects in designs with block size two with initial block (1, 3) *and* (1, 4)

r	U_r	V_r	v	$(3\gamma_1-\beta_1)\,v$	$(\beta_2-3\gamma_2)\,v$	$(\beta_1\gamma_2-\beta_2\gamma_1)$	E.F.
0	1	1	5	−30	75	6	0·6522
1	−2	1	7	0	−91	13	0·5417
2	2	−3	9	171	−126	37	0·5092
3	2	6	11	−473	770	109	0·4866
4	−10	−2	13	520	−1,495	325	0·4688
5	17	−11	15	570	750	964	0·4498
6	−8	33	17	−3,468	3,893	2,857	0·4314
7	−32	−39	19	6,555	−12,578	8,473	0·4144
8	96	−8	21	−3,549	16,926	25,129	0·3985
9	−120	136	23	−14,812	3,565	74,521	0·38[illegible]3
10	−15	−279	25	48,170	−68,450	2,20,996	0·3739
11	390	225	27	−16,312	1,53,441	6,55,381	0·3553
12	−830	325	29	−7,569	−1,34,270	19,43,581	0·3440
13	706	−1,394	31	2,43,939	−2,00,446	57,63,829	..
14	878	2,166	33	−5,57,040	9,31,821	1,70,93,053	..
15	−4,063	−723	35	5,08,690	−15,46,090	5,06,90,692	..
16	6,512	−4,299	37	6,45,724	5,68,505	15,03,26,929	..
17	−2,560	12,913	39	−30,43,435	37,64,670	44,58,05,425	..
18	−13,568	−14,736	41	55,26,103	−1,07,47,330	1,18,17,14,369	..
19	38,032	−5,232	43	−23,20,452	12,97,88,217	3,92,06,89,777	..
20	−45,087	55,441	45	−1,22,71,950	45,10,710	11,62,70,90,308	..
21	−11,634	−1,08,8[illegible]3	47	3,62,77,232	−5,26,37,6[illegible]3	34,48,09,80,829	..
22	1,59,810	79,821	49	−4,52,22,541	10,85,51,170	1,02,25,58,53,141	..
23	−3,24,0[illegible]0	1,43,766	..	..	..	..	..
24	2,53,080	−5,29,250	..	..	..	..	..
25	3,93,265	8,28,325	..	..	..	..	..

$$13vt_i = \sum_{u=1}^{p} (p-u+1)^2 Q_i^{u-1} + \frac{2v(3\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p-1} U_{p-u-1} Q_i^{u-1}$$

$$+ \frac{2v(\beta_2 - 3\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p} V_{p-u} Q_i^{u-1}$$

and the initial values of U's and V's are

$$U_0 = 1, \ U_1 = -1, \ U_2 = 0, \ U_3 = 4, \text{ etc.}$$

$$V_0 = 1, \ V_1 = 1, \ V_2 = -1, \ V_3 = 3, \text{ etc.}$$

The different variances are the same as given in Case 8 *a*.

All the columns as in Case 8 *a* have also been presented in Table X.

4.3. *Case 9—When $k = 2$ and $r = 4$. $v = 2p + 1$, $b = 2v$*

The designs are obtained by developing the initial block of the type (1, 2) and (1, 4) mod v. This class of designs are partially balanced incomplete block designs with p associate classes with the following parameters:

$$\lambda_i = 1 \ (i = 1, \ 3)$$

$$= 0 \ (i = 2, \ 4, \cdots p)$$

$$n_i = 2 \ (i = 1, \cdots p).$$

The solution for t_i is given by

$$10vt_i = \sum_{u=1}^{p} (p-u+1)(p-u+2) Q_i^{u-1}$$

$$+ \frac{2v(\beta_2 - 2\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p-1} U_{p-u-1} Q_i^{u-1}$$

$$+ \frac{2v(2\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} \sum_{u=1}^{p} V_{p-u} Q_i^{u-1}$$

where

$$\beta_1 = -U_{p-2} + 4U_{p-3} - 2U_{p-4} - U_{p-6},$$

$$\gamma_1 = U_{p-1} - U_{p-3},$$

TABLE X

Values of U_r, V_r, $(3\gamma_1 - \beta_1)\,v$, $(\beta_2 - 3\gamma_2)\,v$, $(\beta_1 r_2 - \beta_2 r_1)$ and E.F. which appear in the solutions of treatment effects in designs with block size two with initial blocks (1, 3) *and* (1, 4)

r	u_r	v_r	v	$(3\gamma_1-\beta_1)v$	$(\beta_2-3\gamma_2)\,v$	$(\beta_1\gamma_2-\beta_2\gamma_1)$	E.F.
0	1	1	8	152	−64	21	0·5069
1	−1	1	10	−240	320	64	0·5000
2	0	−1	14	109	182	559	0·4588
3	4	3	16	−2,656	1,808	1,659	0·4498
4	−8	0	20	3,520	6,400	14,592	0·4062
5	7	−5	22	−17,886	3,256	43,273	9·3907
6	9	15	26	−13,260	63,310	3,80,575	0·3627
7	−40	−15	28	76,076	−46,592	11,28,621	0·3501
8	64	−7	32	−2,88,352	4,08,320	99,25,797	..
9	−24	64	34	79,764	−6,21,622	2,94,25,671	..
10	−135	−119	38	−24,62,894	17,89,496	25,87,80,519	..
11	375	81	40	20,20,720	−46,02,880	76,80,29,504	..
12	−440	175	44	−1,43,73,656	33,16,852	6,75,17,56,371	..
13	−124	−629	46	2,29,60,716	−2,34,25,424	20,02,28,23,927	..
14	1,584	896	50	5,70,68,400	−2,87,40,800	1,76,09,31,48,736	..
15	−3,195	−141	..	..	..	..	..
16	2,449	−2,337	..	..	..	..	..
17	3,952	5,665	..	..	..	..	..
18	−16,128	−5,775	..	..	..	..	..
19	24,464	−3,840	..	..	..	..	..
20	−7,055	25,745	..	..	..	..	..
21	−56,721	−46,367	..	..	..	..	..
22	1,48,176	27,679	..	..	..	..	..
23	−1,64,220	75,411	..	..	..	..	..
24	−71,000	−2,51,264	..	..	..	..	..
25	6,46,295	3,40,075	..	..	..	..	..

$$U_u = - U_{u-2} + 4U_{u-3} - U_{u-5} - U_{u-6},$$

$$U_0 = 1,\ U_1 = -1,\ U_2 = -1,\ U_3 = 5,\ \text{etc.}$$

and

$$\beta_2 = - V_{p-1} + 4V_{p-2} - 2V_{p-3} - V_{p-5},$$

$$\gamma_2 = V_p - V_{p-2},$$

$$V_u = - V_{u-2} + 4V_{u-3} - V_{u-5} - V_{u-6},$$

$$V_0 = 1,\ V_1 = 1,\ V_2 = -2,\ V_3 = 2,\ V_4 = 5,\ \text{etc.}$$

The different variances are given by

$$\text{Var}\,(t_i - t_{i \pm u})$$

$$= \frac{1}{5v} \left\{ u(v-u) + \frac{2v(\beta_2 - 2\gamma_2)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} (U_{p-2} - U_{p-u-2}) + \frac{2v(2\gamma_1 - \beta_1)}{(\beta_1\gamma_2 - \beta_2\gamma_1)} (V_{p-1} - V_{p-u-1}) \right\} \sigma^2$$

$$u = 1,\ 2, \cdots p.$$

Tables for U_u, V_u, $(\beta_2 - 2\gamma_2)\,v$, $(2\gamma_1 - \beta_1)\,v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ and E.F. have been calculated for different v's and presented in Table XI.

In order to compare the different designs of the type having the same primary parameters, the following table has been prepared.

The above table clearly shows the E.F. for the initial blocks (1, 2) (1, 4) is greater than with the initial block (1, 2) (1, 3) and E.F. for the initial block (1, 3) (1, 4) is greater than that of (1, 2) (1, 4). In certain cases E.F.'s are same for two different types which means that one case is reducible to the other one.

Table for efficiency factor ($k = 2$, $r = 4$)

Number of treatments	E.F. of the designs with the initial blocks (1, 2) and (1, 3)	E.F. of the designs with the initial blocks (1, 2) and (1, 4)	E.F. of the designs with the initial blocks (1, 3) and (1, 4)
5	0·6250	0·6522	0·6522
6	0·5769	—	—
7	0·5414	0·5417	0·5417
8	0·5096	—	0·5096
9	0·4742	0·5092	0·5092
10	0·4492	—	0·5000
11	0·4246	0·4866	0·4866
12	0·4024	—	—
13	0·3824	0·4576	0·4688
14	0·3642	—	0·4588
15	0·3476	0·4333	0·4498
16	0·3324	—	0·4498
17	0·3184	0·4105	0·4314
18	0·3056	—	—
19	0·2938	0·3843	0·4144
20	0·2828	—	0·4062
21	0·2726	0·3730	0·3984
22	0·2631	—	0·3903
23	0·2542	0·3547	0·3803
24	0·2459	—	—
25	0·2382	0·3378	0·3739
26	0·2309	—	0·3627
27	0·2240	0·3242	0·3553
28	0·2175	—	0·3501
29	0·2107	0·3110	0·3440
30	0·2057	—	—

(—) means designs does't exist.

TABLE XI

Values of U_r, V_r, $(2\gamma_1 - \beta_1)\,v$, $(\beta_2 - 2\gamma_2)\,v$, $(\beta_1\gamma_2 - \beta_2\gamma_1)$ and E.F. which appear in the solutions of treatment effects in designs with block size two with initial blocks (1, 2) *and* (1, 4)

r	u_r	v_r	v	$(2\gamma_1-\beta_1)v$	$(\beta_2-2\gamma_2)v$	$(\beta_1\gamma_2-\beta_2\gamma_1)$	E.F.
0	1	1	5	−5	45	6	0·6522
1	−1	−1	7	−63	14	13	0·5417
2	−1	−2	9	153	−234	37	0·5092
3	5	2	11	11[illegible]	3[illegible]2	109	0·4866
4	−4	5	13	−663	364	313	0·45[illegible]6
5	8	−11	15	1,035	—2,010	905	0·4333
6	24	4	17	527	2,008	2,117	0·4106
7	−12	28	19	−5,111	4,[illegible]28	7,561	0·3843
8	−51	−51	21	6,027	−13,776	21,853	0 3730
9	111	−3	23	8,073	8,326	63,157	0·3547
10	−17	154	25	—33,775	33 950	1,825,25	0·3378
11	−295	−218	27	26,811	−82.728	5,27,50[illegible]	0·3242
12	488	−119	29	70,441	18,212	15,24,529	0·3110
13	128	809	31	−1,97 408	2,45,582	46,13,872	..
14	−1,600	−856	33	86,031	−4,48,338	1,27,33,489	..
15	2,008	−1,064	35	5,19,785	−94,789	3,68,00,465	.
16	1,641	4,057	37	—10,95,237	16 01,656	10,63,55,317	..
17	−8,241	−2,951	39	41,457	−22,05 266	30,73,72,573	..
18	7,503	−7,338	41	34,42,729	16,87,642	88,83,23,221	..
19	12.669	19,434	43	−54,97,077	96,50,576	2,56,73,01,757	..
20	−40,508	−7,667	45	—23,72,535	−96,53.220	6,94,37,89,189	..
21	23,576	−44,771	47	2,11,85,391	−1,55.04,369	21,80,47,15,944	..
22	83.516	88,684	49	−2,52,09,373	5,43,95,831	63,10,54,87,512	..
23	−1,91,512	−2,379	..	..	..	..	..
24	43,793	−2,52,764	..	..	..	..	..
25	4,89,331	3,82,452	..	..	..	..	..

5. How to Use Tables

An illustration with $v = 15$ *and* $k = 3$.

$$\text{For} \quad v = 15,\ p = \frac{v-1}{2} = 7,\ k = r = 3.$$

The solution for t_i can be written with the help of Table I, and is given by the equation:

$$4vt_i = \sum_{u=1}^{7} (8-u)(9-u)\, Q_i^{u-1} - \frac{2v}{\beta_1} \sum_{u=1}^{7} U_{7-u} Q_i^{u-1}$$

where $\beta_1 = -7953$ for $v = 15$.

The value of β_1 has been obtained from Col. II entered against $v = 15$ and substituting the values of $U_6 \cdots U_0$, we have $(4 \times 15 \times 2651)\ t_i = 2651\,\{56Q_i + 42Q_i^1 + 30Q_i^2 + 20Q_i^3 + 12Q_i^4 + 6Q_i^5 + 2Q_i^6\} + 2 \times 5\,\{2296Q_i - 615Q_i^1 + 165Q_i^2 - 44Q_i^3 + 12Q_i^4 - 3Q_i^5 + Q_i^6\}$ or $159060\, t_i = 171416Q_i + 105192Q_i^1 + 81180Q_i^2 + 52580Q_i^3 + 31932Q_i^4 + 15876Q_i^5 + 5312Q_i^6$.

The values of U's have been obtained from Col. I of the table. Thus the above expression gives the solution of t_i which can further be simplified. The different variances can be obtained by using either the formula or directly from the above expression.

6. Summary

In an attempt to get incomplete block designs for each and every number of treatments the problem of analysis of circular designs was introduced by Das and was solved for the designs with the blocks of size two has been solved for plots of sizes three and four. Necessary tables required for obtaining the intra-block analysis readily have also been prepared and presented. A special class of designs with blocks of two plots and having more than one initial block has been investigated. Another class of more efficient type of designs with block size three has also been investigated.

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